Physical origin of the Gouy phase shift by Simin Feng, Herbert G. Winful *Opt. Lett.* **26**, 485-487 (2001)

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For Gaussian beam or TEM_{00} mode,

$$E(r,z) = E_0 \frac{w_0}{w(z)} \exp\left(\frac{-r^2}{w(z)^2} - ikz - ik\frac{r^2}{2R(z)} + i\phi_{\mathbf{G}}(\mathbf{z})\right) \tag{1}$$

Along its propagation direction, a Gaussian beam acquires a phase shift which differs from that for a plane wave with the same optical frequency. This difference is called the *Gouy phase shift*, named after L. G. Gouy who first discovered this phenomenon.

$$\phi_G(z) = -\arctan(z/z_R) \tag{2}$$

where $z_R = \pi w_0^2/\lambda$ is the Rayleigh length and $z = 0$ corresponds to the position of the beam waist.

L. G. Gouy

Although Gouy made his discovery more than 100 years ago, efforts are still being made to provide a simple and satisfying physical interpretation of this phase anomaly.

In this letter, the authors provide a simple intuitive explanation of the physical origin of the Gouy phase shift.

Consider a monochromatic wave of frequency ω and wavenumber $k = \omega/c$ along z direction.

If it is an *infinite plane wave*, the momentum has no transverse components. A finite beam, however, consists transverse components of momentum. The wavenumber can be written as:

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{3}$$

Since the wave-vector components are finitely spread , it is appropriate to deal with expectation values (or simply averages) defined by:

$$\langle \xi \rangle = \frac{\int_{-\infty}^{\infty} \xi |f(\xi)|^2 d\xi}{\int_{-\infty}^{\infty} |f(\xi)|^2 d\xi} \tag{4}$$

where $f(\xi)$ is the distribution of the variable ξ

The effective axial propagation constant can be defined as

$$\bar{k_z} = \frac{\langle k_z^2 \rangle}{k} = k - \frac{\langle k_x^2 \rangle}{k} - \frac{\langle k_y^2 \rangle}{k} \tag{5}$$

which is associated with the overall propagation phase $\phi(z)$ through $\bar{k_z} = \partial\phi(z)/\partial z$. By integrating with respect to z we get

$$\phi(z) = kz - \frac{1}{k}\int^z \{\langle k_x^2 \rangle + \langle k_y^2 \rangle\} dz \tag{6}$$

Here the first term stands for phase evolution of infinite plane wave, and the second gives rise to the Gouy phase shift

$$\phi_G = -\frac{1}{k}\int^z \{\langle k_x^2 \rangle + \langle k_y^2 \rangle\} dz \tag{7}$$

$$\phi_G = -\frac{1}{k}\int^z \{\langle k_x^2\rangle + \langle k_y^2\rangle\} dz \tag{8}$$

Hence, the Gouy shift is due to the *transverse momentum spread*. It can be well formulated since we know transverse distribution exactly. For Gaussian beam, transverse distribution is given by:

$$f(x,y) = \sqrt{\frac{2}{\pi}}\frac{1}{w(z)}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right] \tag{9}$$

Here $w^2(z) = w_0^2\left[1+\left(\frac{z}{z_R}\right)^2\right]$ is beam radius and w_0 is the minimum spot size at $z=0$. The Rayleigh length z_R is defined by $z_R = \pi w_0^2/\lambda$

The distribution of transverse wavevector components is given by the Fourier transform.

$$
\begin{aligned}
F(k_x, k_y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-ik_x x}e^{-ik_y y}dxdy \\
&= \sqrt{\frac{1}{2\pi^3}}\frac{1}{w(z)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]e^{-ik_x x}e^{-ik_y y}dxdy \\
&= \frac{w(z)}{\sqrt{2\pi}}\exp\left[-\frac{w^2(z)}{4}(k_x^2+k_y^2)\right]
\end{aligned}
\tag{10}
$$

We can see this is also Gaussian and centered about $k_x = k_y = 0$. And both of the distribution are normalized such that $\int\int f(x,y)dxdy = \int\int F(k_x,k_y)dk_x dk_y = 1$

Now, we have

$$
\begin{aligned}
\langle k_x^2 \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_x^2 |F(k_x, k_y)|^2 dk_x dk_y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_x^2 \frac{w^2(z)}{2\pi} \exp\left[-\frac{w^2(z)}{2}(k_x^2 + k_y^2) \right] dk_x dk_y \\
&= \frac{w^2(z)}{2\pi} \int_{-\infty}^{\infty} k_x^2 \exp\left[-\frac{w^2(z)}{2} k_x^2 \right] dk_x \int_{-\infty}^{\infty} \exp\left[-\frac{w^2(z)}{2} k_y^2 \right] dk_y \\
&= \frac{1}{w^2(z)} = \langle k_y^2 \rangle
\end{aligned}
$$

The Gouy phase shift for the Gaussian beam is then:

$$
\phi_G = \frac{1}{k} \int^{z} \{ \langle k_x^2 \rangle + \langle k_y^2 \rangle \} dz = -\frac{2}{k} \int^{z} \frac{1}{w^2(z)} dz = -\arctan(z/z_R) \qquad (11)
$$

We now show that our equation predicts the phase anomaly not only for fundamental Gaussian beams but also for higher-order transverse modes and hence is valid for arbitrary field distributions.

One complete set of transverse modes is described by Hermite-Gaussian beams.

$$f_{mn}(x, y) = C_{mn} \frac{\sqrt{2}}{w(z)} \Theta_m \left[\frac{\sqrt{2}x}{w(z)} \right] \Theta_n \left[\frac{\sqrt{2}y}{w(z)} \right] \tag{12}$$

where $\Theta_m(\xi) = H_m(\xi) \exp(-\xi^2/2)$ is the Hermite-Gaussian of mth order and $C_{mn} = \left(\frac{1}{\pi 2^{m+n} m! n!} \right)^{1/2}$ is normalization coefficient.

The Fourier transform of Hermite-Gaussian distribution is

$$F_{mn}(k_x, k_y) = (-i)^{m+n} C_{mn} \frac{w(z)}{\sqrt{2}} \Theta_m \left[\frac{w(z) k_x}{\sqrt{2}} \right] \Theta_n \left[\frac{w(z) k_y}{\sqrt{2}} \right] \tag{13}$$

Utilizing the recursion relation $H_{n+1} - 2\xi H_n + 2nH_{n-1} = 0$ for the Hermite polynomials, one can derive the expectation values.

$$\langle k_x^2 \rangle_{mn} = \frac{2}{w^2(z)} \left(m + \frac{1}{2} \right)$$
$$\langle k_y^2 \rangle_{mn} = \frac{2}{w^2(z)} \left(n + \frac{1}{2} \right)$$

Hence, Gouy shift for TEM_{mn} modes are formulated by:

$$\phi_{G,mn}(z) = -(m+n+1) \arctan(z/z_R) \tag{14}$$

Conclusion

- A general expression and physical explanation of the Gouy phase shift is discussed, by showing that the Gouy phase can be derived from the transverse momenta.

- Gouy shift of finite beams stems from transverse spatial confinement, which consequently causes transverse momentum component.

- This approach is valid for higher transverse modes, as well as for fundamental transverse (Gaussian) mode.