Torque Detection using Brownian Fluctuations

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Abstract

We report the statistical analysis of the movement of a submicron particle confined in a harmonic potential in the presence of a torque. The absolute value of the torque can be found from the auto- and cross-correlation functions of the particle's coordinates. We experimentally prove this analysis by detecting the torque produced onto an optically trapped particle by an optical beam with orbital angular momentum.

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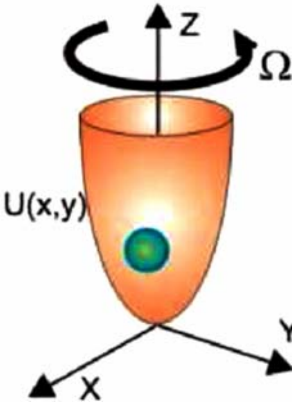
Theory

We consider a sphere of mass *m* and radius *R* suspended in a liquid medium and confined within a harmonic potential well, where it moves randomly due to the thermal excitation



1. An external torque is exerted on the sphere :

balance between the torque applied to the sphere and the drag torque

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the friction the sphere rotates around the *z* axis with a constant angular velocity Ω

$$\boldsymbol{\tau}_{\text{drag}} = \mathbf{r} \times \mathbf{F}_{\text{drag}} = \gamma \mathbf{r} \times \mathbf{v}$$
$$= \gamma \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\Omega})$$

$\mathbf{r}$: sphere's position
$\mathbf{v}$: linear velocity
$\gamma = 6\pi R\eta$: friction coefficient
η : viscosity

time average of the torque

$$\langle \tau \rangle = \gamma \langle \mathbf{r} \times (\mathbf{r} \times \Omega) \rangle = \gamma \langle \Omega r^2 \rangle = \gamma \Omega \langle r^2 \rangle$$

2. The Einstein-Ornstein-Uhlenbeck equations [24] for the Brownian motion of the sphere in the plane perpendicular to the rotation axis can now be presented as:

$$\gamma \frac{dx(t)}{dt} + kx(t) + \gamma\Omega y(t) = \sqrt{2k_B T\gamma}\,\eta_x(t), \qquad (1)$$

$$\gamma \frac{dy(t)}{dt} + ky(t) - \gamma\Omega x(t) = \sqrt{2k_B T\gamma}\,\eta_y(t), \qquad (2)$$

k : force constant of the harmonic oscillator

$\sqrt{2k_B T\gamma}\,\eta_x(t)$ and $\sqrt{2k_B T\gamma}\,\eta_y(t)$: are two independent white Gaussian random processes that represent the Brownian forces at temperature *T* in the *x* and *y* direction

$m\frac{d^2y(t)}{dt^2}$: neglect all inertial terms (low Reynolds number regime)

[24] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

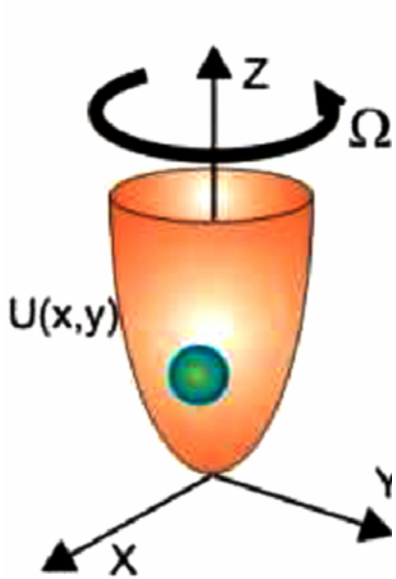
The terms $+\gamma\Omega y(t)$ and $-\gamma\Omega x(t)$ introduce a coupling between the equations.

The auto- and cross-correlation functions for the movement of the sphere along the *x* and *y* directions are given by:

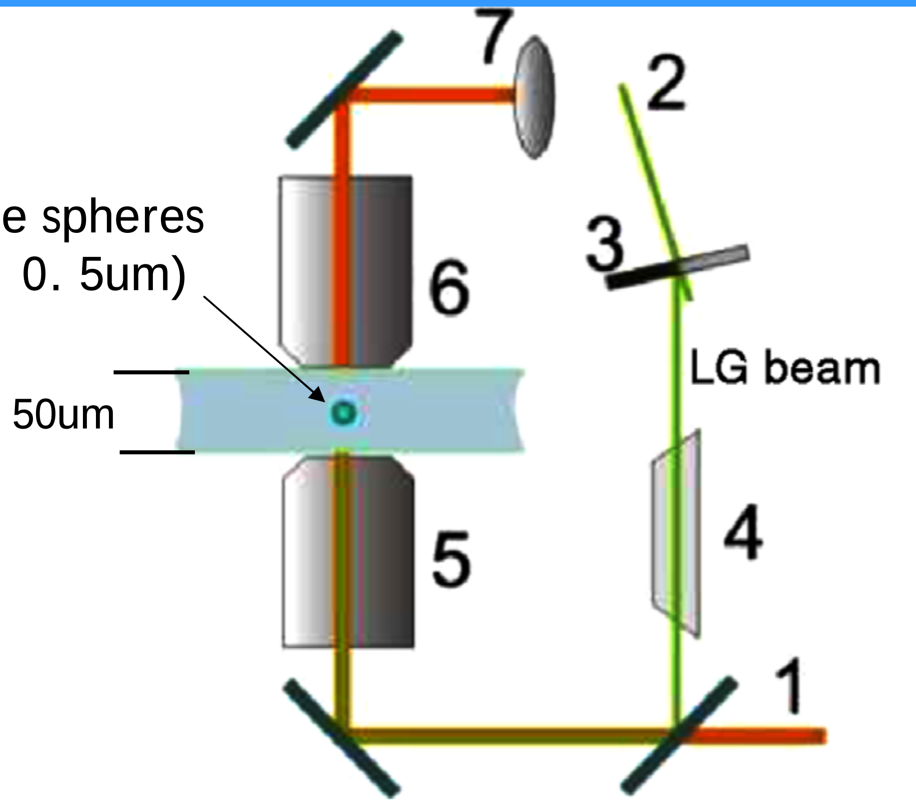
$$\langle x(t)x(t+\Delta t)\rangle = \langle y(t)y(t+\Delta t)\rangle = \frac{k_B T}{k} e^{-k|\Delta t|/\gamma} \cos(\Omega \Delta t), \qquad (3)$$

$$\langle x(t)y(t+\Delta t)\rangle = \frac{k_B T}{k} e^{-k|\Delta t|/\gamma} \sin(\Omega \Delta t). \qquad (4)$$

Experimental setup

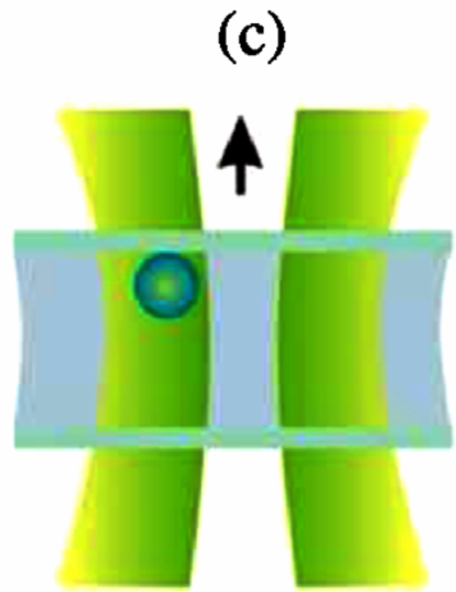


Harmonic potential well $U(x, y)=k/2(X^2+Y^2)$
(k is the restoring force constant or stiffness of the harmonic oscillator) with a Brownian particle inside and with an external torque acting on the particle.

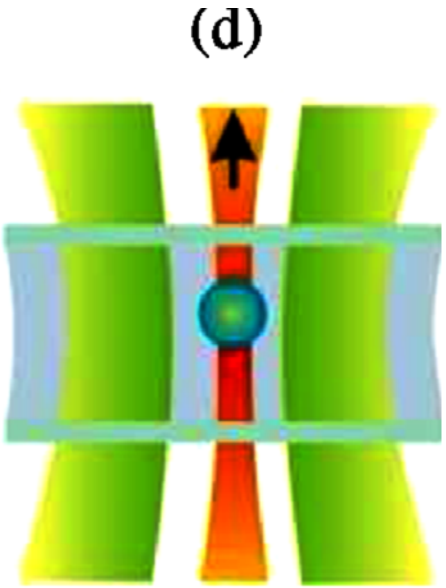


1—trapping 785 nm laser beam (electric field gradient forces)
2—532 nm beam (orbital angular momentum)
3—holographic mask
4—Dove prism
5—100 *1.3*NA* objective
6—collimating 40 objective
7—quadrant photodetector (QPD)

Position of the sphere in the chamber

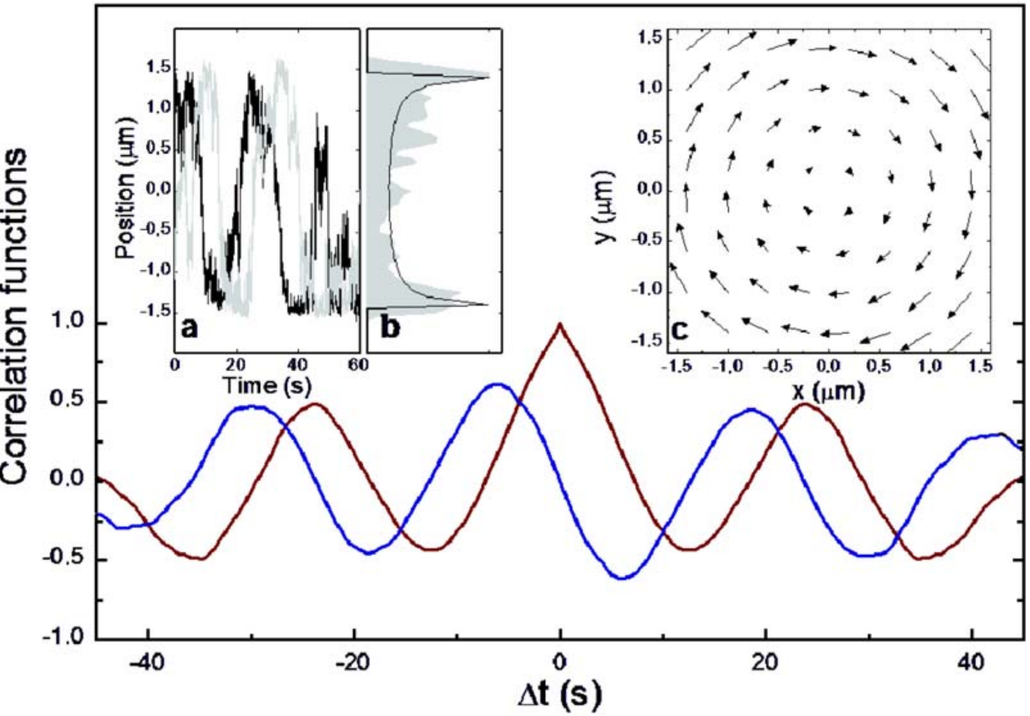


532 nm Laguerre-Gaussian(LG) propagates in the chamber.



both the 532 nm and the 785 nm beams propagate in the chamber.

The trap force constant k is low (0.9 fN/μm)



rotation period : $T_r = 20 \pm 5$ s

orbital radius : $r = 1.4 \pm 0.3\ \mu$m

torque : $\gamma \Omega r^2 = 3.9 \pm 1.8 \times 10^{-21}$ Nm

friction coefficient $\gamma = 4 \pm 1 \times 10^{-8}$ N s/m

FIG. 2 (color online). Experimental unbiased auto- and cross-correlation functions in the presence of the torque induced by a LG beam with $l = +10$. The trap force constant k is low enough (0.9 fN/μm) not to significantly influence the rotational motion of the sphere. The continuous lines show the mean values obtained using one series of data acquisition (acquisition time 60 s, sampling rate $f_s = 1$ kHz). In the insets: (a) time traces for the x (black) and y (gray) coordinates; (b) histogram of the x coordinate; (c) vector force field acting on the particle in the xy plane.

The trap force constant *k* is high (16 fN/μm,100uW))

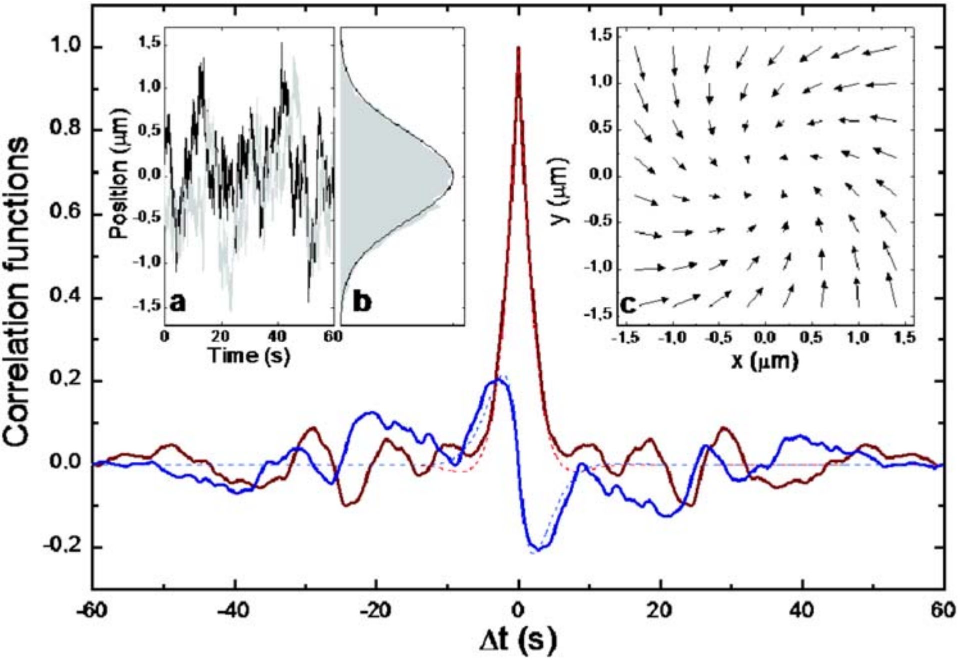


FIG. 3 (color online). Experimental auto- and cross-correlation functions in presence of the torque induced by a LG beam with $l = +10$. The trap force constant k is high enough (16 fN/μm) to confine the sphere. The continuous lines show the mean values obtained using five series of data acquisition (acquisition time 60 s, sampling rate $f_s = 1$ kHz). The dotted lines show the fitting to the theoretical shape (the fitting was made on the central part of the curve for $\Delta t = [-2 \text{ s}, 2 \text{ s}]$). In the insets: (a) time traces for the x (black) and y (gray) coordinates; (b) histogram of the x coordinate and in black the fitting to a Gaussian distribution; (c) vector force field acting on the particle in the xy plane.

the sphere is more confined in the center of the trapping beam and does not display a rotational motion,

Fourier analysis of the experimental traces does not show the presence of the torque existing in the system.

near t 0 --->
the torque produced by the orbital angular momentum of the LG beam still affects the Brownian trajectories.

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Torque : $4.9 \pm 0.7 \times 10^{-21}$ Nm

The trap force constant *k* is high (insert Dove prism)

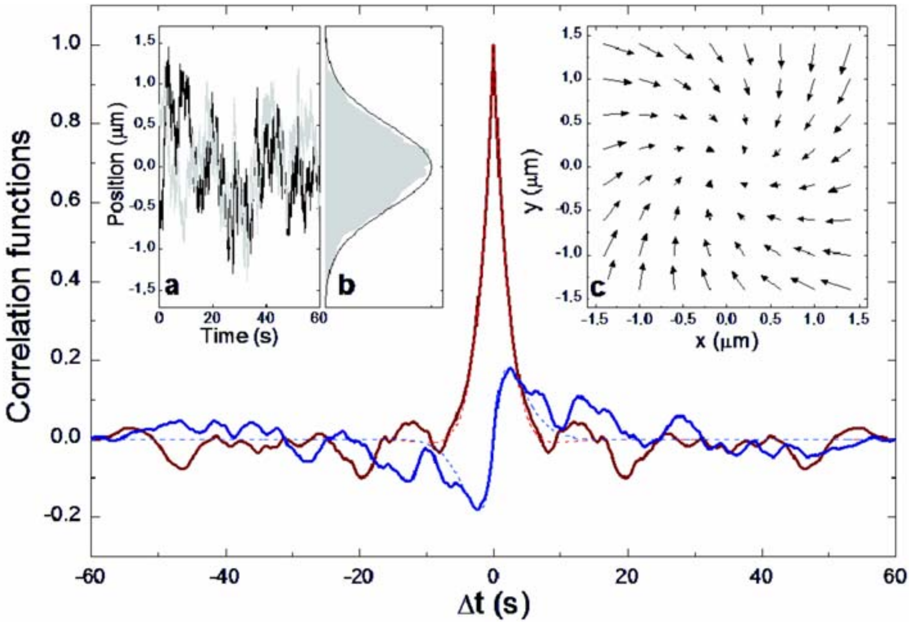

FIG. 4 The same as in Fig. 2 when the torque is produced by a LG with *l* = -10.

Torque : $3.9 \pm 0.8 \times 10^{-21}$ Nm

conclusion

1. The value of the torques measured ($4*10^{-21}$Nm)

2. A detailed analysis of the Brownian fluctuations of a particle trapped in a harmonic potential may be a starting point to build new tools for the measurement of torque in micrometric systems

3. Make it possible to study how the torque exerted by a certain source varies in the presence of a controlled mechanical load

Stokes's drag

[edi

The equation for **viscous resistance** or **linear drag** is appropriate for small objects or particles moving through a fluid at relatively slow speeds where there is no turbulence (i.e. low Reynolds number, $R_e < 1$).[1] In this case, the force of drag is approximately proportional to velocity, but opposite in direction. [1] The equation for viscous resistance is:

$$\mathbf{F}_d = -b\mathbf{v}$$

where:

b is a constant that depends on the properties of the fluid and the dimensions of the object, and

v is the velocity of the object.

When an object falls from rest, its velocity will be

$$v(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right)$$

which asymptotically approaches the terminal velocity $v_t = mg / b$. For a given b, heavier objects fall faster.

For the special case of small spherical objects moving slowly through a viscous fluid (and thus at small Reynolds number), George Gabriel Stokes derived an expression for the drag constant,

$$b = 6\pi\eta r$$

where:

r is the Stokes radius of the particle, and η is the fluid viscosity.

For example, consider a small sphere with radius r = 0.5 micrometre (diameter = 1.0 μm) moving through water at a velocity **v** of 10 μm/s. Using 10^{-3} Pa·s as the dynamic viscosity of water in SI units, we find a drag force of 0.28 pN. This is about the drag force that a bacterium experiences as it swims through water.

레이놀즈수 (－數 Reynolds number)

물체를 지나는 유체의 흐름 또는 유로 속에서의 유체흐름의 관성력과 점성력의 크기의 비를 알아보는 데 있어서 지표가 되는 무차원수.

설　　　명

물체를 지나는 유체의 흐름 또는 유로(流路) 속에서의 유체흐름의 관성력(관성저항)과 점성력의 크기의 비를 알아보는 데 있어서 지표가 되는 무차원수. O. 레이놀즈에 의하여 도입되었으며, 보통 또는 로 나타낸다. 흐름 속에 있는 물체의 대표적 길이를 (원통 속의 흐름의 경우에는 원통의 지름, 흐름 속에 球가 있는 경우에는 그 구의 반지름), 유속을 , 유체밀도를 , 점성률을 라 하면,

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로 정의된다(여기서 = / 는 운동점성계수). 흐름 상태는 에 의해 크게 달라지므로, 는 흐름의 특징을 정해지게 하는 데 가장 중요한 조건이 된다. 가 작은 동안은 정류(整流)상태이었던 흐름도, 가 임계값[臨界値(임계치):臨界(임계)레이놀즈수라 한다]을 넘게 되면 불규칙적으로 변동하는 난류(亂流)로 변하게 된다.

Power spectral density

[edit]

The above definitions of energy spectral density require that the Fourier transforms of the signals exist, that is, that the signals are square-integrable or square-summable. An often more useful alternative is the **power spectral density** (PSD), which describes how the power of a signal or time series is distributed with frequency. Here power can be the actual physical power, or more often, for convenience with abstract signals, can be defined as the squared value of the signal, that is, as the actual power if the signal was a voltage applied to a 1-ohm load. This instantaneous power (the mean or expected value of which is the average power) is then given by:

$$P = s(t)^2 \ .$$

Since a signal with nonzero average power is not square integrable, the Fourier transforms do not exist in this case. Fortunately, the Wiener–Khinchin theorem provides a simple alternative. The PSD is the Fourier transform of the autocorrelation function, $R(\tau)$, of the signal if the signal can be treated as a stationary random process.

This results in the formula,

$$S(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-2\,\pi\,i\,f\,\tau}\, d\tau.$$

The power of the signal in a given frequency band can be calculated by integrating over positive and negative frequencies,

$$P = \int_{F_1}^{F_2} S(f)\, df + \int_{-F_2}^{-F_1} S(f)\, df.$$

The power spectral density of a signal exists if and only if the signal is a wide-sense stationary process. If the signal is not stationary, then the autocorrelation function must be a function of two variables, so no PSD exists, but similar techniques may be used to estimate a time-varying spectral density.